Journal of Maritime Research, Vol. V. No. 1, pp. 65-78, 2008

ISSN: 1697-4840

A DERIVATION OF HIGH-FREQUENCY ASYMPTOTIC VALUES OF 3D ADDED MASS AND DAMPING BASED ON PROPERTIES OF THE CUMMINS' EQUATION

T. Perez[1,3] and T. I. Fossen[2]

Received 5 September 2007; received in revised form 15 September 2007; accepted 15 March 2008

ABSTRACT

This brief paper provides a novel derivation of the known asymptotic values of three-dimensional (3D) added mass and damping of marine structures in waves. The derivation is based on the properties of the convolution terms in the Cummins's Equation as derived by Ogilvie. The new derivation is simple, and no approximations or series expansions are made. The results follow directly from the relative degree and low-frequency asymptotic properties of the rational representation of the convolution terms in the frequency domain. As an application, the extrapolation of damping values at high frequencies for the computation of retardation functions is also discussed.

INTRODUCTION

The ability to predict ship responses and loads in waves is an important tool in the design of marine structures and motion control systems. One method of constructing time-domain models consists of using the data generated by the hydrodynamic codes to compute the different elements of the so called Cummins's equation of ship motion (Cummins, 1962). This equation contains some convolution terms which describe fluid memory effects. The kernel of the convolution terms (impulse responses) can be computed from the frequency-dependent potential damping. For

[1] Centre for Complex Dynamic Systems and Control—CDSC (tristanp@ieee.org), The University of Newcastle, Callaghan, NSW 2308, Australia. [2] Department of Engineering Cybernetics, Norwegian University of Science and Technology—NTNU, Norway. [3] Corresponding author (tristan.perez@ntnu.no).

these computations to be accurate, it is often necessary to compute the hydrodynamic damping for very high frequencies-even when at these high frequencies the response of the vessel is negligible.

When frequency-domain hydrodynamic codes are used to compute added mass and damping; there is a limitation on the highest frequency that can be reached, and this may limit the accuracy of the retardation function. Indeed, in order to have accurate results of added mass and damping, it is recommended to use a panelling size such that the panel characteristic length is less than $1/8^{th}$ of the wave length (Faltinsen, 1993). This implies that in order to reach high frequencies, the size of the panels need to be reduced–with the consequence of a very large number of computations and time. One way to alleviate this problem consists of extrapolating the potential damping based on its high-frequency asymptotic trend.

The asymptotic values of damping and added mass have been discussed in the literature in terms of expansions and the Kramer-Kronig relations (Greenhow, 1986). The contribution of this paper is to provide a much simpler and more elegant derivation of such asymptotic values based on the properties of the convolution terms in the Cummins's Equation. The results obtained show that in the limit as the frequency tends to infinity, both damping and added mass tail towards their asymptotic values at a rate of ω^{-2}, which, as discussed by Damaren (2000), seems at odds with the traditional results of the hydrodynamic literature, but coincide as the limit is taken to high frequencies.

LINEAR TIME-DOMAIN EQUATIONS OF MOTION

For a marine structure at zero forward speed a linear time-domain model takes the following form (Cummins, 1962)[a]:

$$[\mathbf{M}+\mathbf{A}]\,\ddot{\boldsymbol{\xi}}+\int_0^t \mathbf{K}(t-\tau')\,\dot{\boldsymbol{\xi}}(\tau')\,d\tau'+\mathbf{C}\,\boldsymbol{\xi}=\boldsymbol{\tau} \qquad (1)$$

Where the motion and force variables are

$$\boldsymbol{\xi}=\begin{bmatrix}\xi_1\\ \vdots\\ \xi_6\end{bmatrix},\quad \boldsymbol{\tau}=\begin{bmatrix}\tau_1\\ \vdots\\ \tau_6\end{bmatrix}$$

Note that **bold symbols** represent vector and matrix quantities.

[a] The lower limit in the integral term id (1) as derived by Cummins (1962) is $-\infty$; however, under the assumption of causality, K(t)=0 for $t<0$, the a lower limit of 0 yields the same value for the convolution term in (1).

These variables represent the motion and excitation forces of the structure due to the waves with respect to a coordinate system fixed to the mean free surface. The matrix **A** is a constant added-mass matrix. The matrix **K**(t) is a retardation function; i.e., its entries are impulse responses from the velocities to the part of the radiation forces that represent the fluid-memory effects.

Ogilvie, (1964), took the Fourier transform of (1) obtained the following relations with the frequency-dependant added mass and damping:

$$\mathbf{A}(\omega) = \mathbf{A} - \frac{1}{\omega}\int_0^\infty \mathbf{K}(t)\sin(\omega t)\, dt \tag{2}$$

$$\mathbf{B}(\omega) = \int_0^\infty \mathbf{K}(t)\cos(\omega t)\, dt \tag{3}$$

From a theoretical perspective, the retardation functions can be evaluated from the frequency-dependent damping:

$$\mathbf{K}(t) = \frac{2}{\pi}\int_0^\infty \mathbf{B}(\omega)\cos(\omega\, t)\, d\omega \tag{4}$$

When a hydrodynamic code based on panel methods is used to compute the potential damping for different frequencies, there are limitations on the highest frequency for which the damping can be computed. In order to have accurate results of added mass and damping, it is recommended to use a panelling size such that the panel characteristic length is less than $1/8^{\text{th}}$ of the wave length (Faltinsen, 1993). This implies that in order to reach high frequencies, the size of the panels need to be reduced–with the consequence of a very large number of computations and time.

Because of the finite-frequency Ω, there is an error in the computed retardation function:

$$\begin{aligned}\hat{\mathbf{K}}(t) &= \frac{2}{\pi}\int_0^\Omega \mathbf{B}(\omega)\cos(\omega t)d\omega \\ \mathbf{Error}(t) &= \frac{2}{\pi}\int_\Omega^\infty \mathbf{B}(\omega)\cos(\omega t)d\omega\end{aligned} \tag{5}$$

To minimize the error, it is necessary to increase the frequency **Ω** as much as possible. This can be achieved by considering the high-frequency trend of the damping.

PROPERTIES OF THE CONVOLUTION TERMS

It also follows from Ogilvie (1964) that[b]

$$\mathbf{K}(j\omega) = \mathbf{B}(\omega) + j\omega\,[\mathbf{A}(\omega) - \mathbf{A}(\infty)] \tag{6}$$

and also

$$\lim_{\omega \to 0} \mathbf{K}(j\omega) = \mathbf{0} \tag{7}$$

$$\lim_{\omega \to \infty} \mathbf{K}(j\omega) = \mathbf{0} \tag{8}$$

$$\mathbf{K}(t = 0^{+}) \neq \mathbf{0} \tag{9}$$

$$\lim_{t \to \infty} \mathbf{K}(t) = \mathbf{0} \tag{10}$$

Equations (7) and (8) establish the asymptotic value at low and high frequencies respectively. Equation (9) establishes that the initial time value of the retardation function is different from zero. Equation (10) establishes the bounded-input bounded-output stability of the convolution terms. For further details about these properties, see the APPENDIX.

Due to the linearity of the convolution terms, these can be represented in the frequency domain by a rational function:

$$K_{mn}(j\omega) = \frac{P_{mn}(j\omega)}{Q_{mn}(j\omega)} = \frac{b_p(j\omega)^p + b_{p-1}(j\omega)^{p-1} + \cdots + b_1(j\omega)}{(j\omega)^q + a_{q-1}(j\omega)^{q-1} + \cdots + a_1(j\omega) + a_0} \tag{11}$$

Note that in (11), we have omitted the constant b_0 term in the numerator polynomial so as to account for the fact that the convolution terms are zero at $\omega = 0$ –*c.f.* (7).

Equation (8) indicates that (11) must be strictly proper; i.e., the relative degree must be greater than zero (the relative degree is the difference between the degree of the denominator minus the degree of the numerator). Equation (9) has further implications on the relative degree of (11). Indeed, condition (9) follows from (2):

[b] The reader is reminded that we are considering the case of zero forward speed; and therefore $B(\infty)=0$. We will comment on the extensions to forward speed in a later section.

$$\lim_{t\to 0^+}\mathbf{K}(t)=\lim_{t\to 0^+}\frac{2}{\pi}\int_0^\infty \mathbf{B}(\omega)\cos(\omega\, t)\, d\omega=\frac{2}{\pi}\int_0^\infty \mathbf{B}(\omega)\, d\omega\neq \mathbf{0} \tag{12}$$

Note that regularity conditions are satisfied for the exchange of limit and integration; and the last equality follows from energy considerations, which establish that the diagonal terms of $\mathbf{B}(\omega)$; namely $B_{ii}(\omega)> 0$–see Faltinsen (1990).

From the initial value theorem of the Laplace Transform, it follows that

$$K_{mn}(t=0^+)=\lim_{s\to\infty} s\, K_{mn}(s)=\lim_{s\to\infty}\frac{b_p s^{p+1}}{s^q}=b_p \quad (if\;\; q=p+1) \tag{13}$$

which will be different from zero if and only if $p+1=q$. Therefore, the relative degree of (11) must be equal to 1 whenever $K_{mm}(t=0^+)\neq 0$ (the integral of the off-diagonal terms–those which are not uniformly zero due to symmetry of the structure–could be zero; but it is not so in most practical cases.).

HIGH-FREQUENCY ASYMPTOTIC VALUES OF DAMPING AND ADDED MASS

We now show the main contribution of the paper. Because of the rational representation (11)) of the frequency response of the convolution terms (6)), the following relations hold for the damping and added mass:

$$B_{mn}(\omega)=\mathrm{Re}\{K_{mn}(j\omega)\}=\mathrm{Re}\left\{\frac{P_{mn}(j\omega)}{Q_{mn}(j\omega)}\right\}=\frac{\mathrm{Re}\{P_{mn}(j\omega)Q_{mn}(-j\omega)\}}{Q_{mn}(j\omega)Q_{mn}(-j\omega)} \tag{14}$$

$$\omega[A_{mn}(\omega)-A_{mn}(\infty)]=\mathrm{Im}\{K_{mn}(j\omega)\}=\mathrm{Im}\left\{\frac{P_{mn}(j\omega)}{Q_{mn}(j\omega)}\right\}=\frac{\mathrm{Im}\{P_{mn}(j\omega)Q_{mn}(-j\omega)\}}{Q_{mn}(j\omega)Q_{mn}(-j\omega)} \tag{15}$$

From (11), it follows that

$$Q_{mn}(-j\omega)=(-1)^q(j\omega)^q+a_{q-1}(-1)^{q-1}(j\omega)^{q-1}+\cdots+(-1)q_1(j\omega)+a_0,$$

and then

$$\begin{aligned}Q_{mn}(j\omega)\,Q_{mn}(-j\omega)&=(-1)^q(j\omega)^{2q}+a_{q-1}(-1)^{q-1}(j\omega)^{2q-1}+\cdots+(j\omega)^q a_0\\&\quad+a_{q-1}(-1)^q(j\omega)^{3q-1}+\ldots+a_{q-1}(j\omega)^{q-1}a_0+\ldots+a_0^{\,2}.\end{aligned} \tag{16}$$

Similarly,

$$P_{mn}(j\omega)\,Q_{mn}(-j\omega) = b_p(-1)^q(j\omega)^{q+p} + b_p a_{q-1}(-1)^{q-1}(j\omega)^{q+p-1} + \cdots + b_p(j\omega)^p a_0 + \ldots + b_1 a_0(j\omega).$$

From the relative degree constraint ($p = q-1$), the latter becomes

$$P_{mn}(j\omega)\,Q_{mn}(-j\omega) = b_p(-1)^q(j\omega)^{2q-1} + b_p a_{q-1}(-1)^{q-1}(j\omega)^{2q-2} + \cdots + b_{q-1}(j\omega)^{q-1} a_0 + \ldots + b_1 a_0(j\omega). \quad (17)$$

Using the properties of the power of the imaginary unit:

$$j^m = \begin{cases} (-1)^{\frac{m}{2}} & m-even \\ (-1)^{\frac{m-1}{2}}\, j & m-odd \end{cases} \quad (18)$$

on (16)) and (17)), it follows that in the limit as $\omega \to \infty$,

$$B_{mn}(\omega) \quad \to \quad \frac{b_p a_{q-1}(-1)^{q-1}(j\omega)^{2q-2}}{(-1)^q(j\omega)^{2q}} = \frac{b_p a_{q-1}}{\omega^2} := \frac{\beta_{mn}}{\omega^2} \quad (19)$$

$$j[A_{mn}(\omega) - A_{mn}(\infty)] \to \frac{b_p(-1)^q(j\omega)^{2q-1}}{\omega(-1)^q(j\omega)^{2q}} = j\frac{-b_p}{\omega^2} := j\frac{\alpha_{mn}}{\omega^2} \quad (20)$$

Since we are taking the limit as $\omega \to \infty$, the results above are obtained by considering the term of highest order in (16) that is real and the corresponding real and imaginary highest order term in (17).

The results (19) and (20) establish that, at high frequencies, the damping and the added mass tend to their asymptotic values at a rate of ω^{-2} provided that the integral of the damping with respect to the frequency is different from zero (which gives the relative degree 1); and this applies to the diagonal as well as to the off-diagonal terms.

These results are in agreement with those previously published in the literature of hydrodynamics–for example Greenhow (1986) has shown via series expansions that

$$B_{mn}(\omega) \quad \rightarrow \frac{\beta'_{mn}}{\omega^2} + \frac{\beta''_{mn}}{\omega^4} \quad as \quad \omega \rightarrow \infty ,$$

$$[A_{mn}(\omega) - A_{mn}(\infty)] \rightarrow \frac{\alpha'_{mn}}{\omega^2} + \frac{\alpha''_{mn}}{\omega^4} \quad as \quad \omega \rightarrow \infty .$$

But these are dominated by the terms proportional to ω^{-2} as the frequency increases–and therefore revert to the results of (19) and (20).

It is also interesting to note from (20), that the sign of $A_{mn}(\omega) - A_{mn}(\infty)$ is opposite that that of

$$b_p = \frac{2}{\pi} \int_0^{\infty} B_{mn}(\omega)\, dt .$$

Therefore, for example, for the diagonal terms which are known to be positive ($B_{nn}(\omega) > 0$), $A_{nn}(\omega) \rightarrow A_{nn}(\infty)$ always from below as $\omega \rightarrow \infty$.

The derivation of the above results is simple and elegant. It is based solely on the interpretation of the properties of the convolution terms discussed in the APPENDIX. In particular the property of relative degree being equal to 1 plays a key role—which holds whenever the integral of the damping with respect to the frequency is different from zero. Furthermore, the derivation makes no use of series expansions as the results previously presented in the literature—see Greenhow (1986) and references therein.

ILLUSTRATION EXAMPLE: MODERN CONTAINERSHIP

As an example of application, we can consider the vertical plane motion of a modern container ship. The hydrodynamic coefficients were computed with WAMIT, and the panel sized were dimensioned so as to compute up to a frequency of 2.5 rad/s—at which there is no appreciable response.

Figure 1 shows the potential damping computed by the hydrodynamic code (in solid line), and the extrapolation at a rate of ω^{-2} (dashed line). As we can appreciate, for couplings 33 and 55 the ω^{-2} tail is a very good approximation. However, for the couplings 35 and 53 the approximation is not as good. For these couplings, it would be necessary to increase the frequency of the computations more, or, alternatively, to use the higher order expansion by Greenhow (1986) ($\beta' \omega^{-2} + \beta'' \omega^{-4}$), which is also shown in Figure 1. Figure 2 show the plots in logarithmic scale (both axis) so as better appreciate the trends.

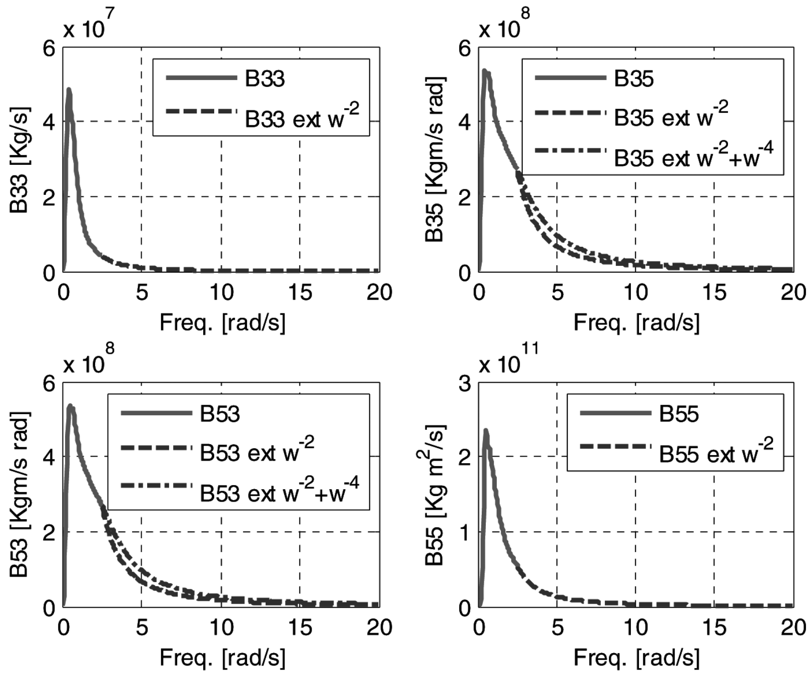

Figure 1 Potential damping linear scale. Solid–computed with a hydrodynamic code. Dashed–extrapolation at a rate of ω^{-2}.

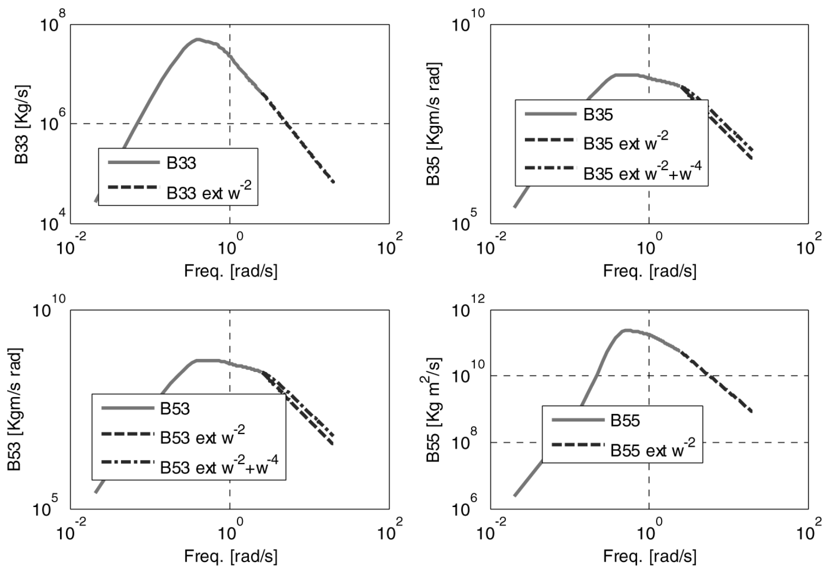

Figure 2 Potential damping log-log scale. Solid–computed with a hydrodynamic code. Dashed–extrapolation at a rate of ω^{-2}.

Figures Figure 3, Figure 4, and Figure 5 show the retardation functions computed using the damping with the extrapolations in (4). The solid lines correspond to the damping computed up to 2.5rad/s only; whereas the dashed lines correspond to the damping extrapolated with a tail proportional to ω^{-2} and for the 35 and 53 with $\beta' \omega^{-2} + \beta'' \omega^{-4}$.

As we can see, in all cases there is a smearing of the error when considering the damping over the higher frequencies. Note also, the error in the initial value of the retardation; which comes from neglecting the area under the tails.

The use of the retardation functions computed from a damping up to a low frequency can give rise to unnecessary dynamics if the retardation functions in used for time-domain identification for convolution replacement–for example, using the method recently proposed by Kristiansen, et al. (2005). Therefore, the results discussed here have application to this field.

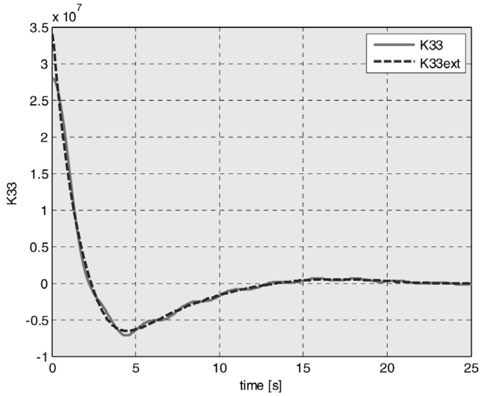

Figure 3. Retardation function 33. Solid–omputed from damping without extrapolation. Dashed–computed from damping with extrapolation ω^{-2}.

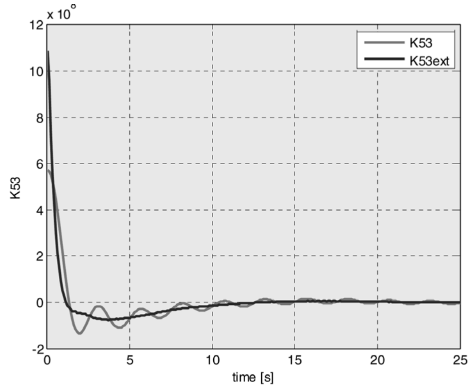

Figure 4. Retardation function 35. Solid–computed from damping without extrapolation. Dashed–computed from damping with extrapolaton $\beta'\omega^{-2}+\beta''\omega^{-4}$).

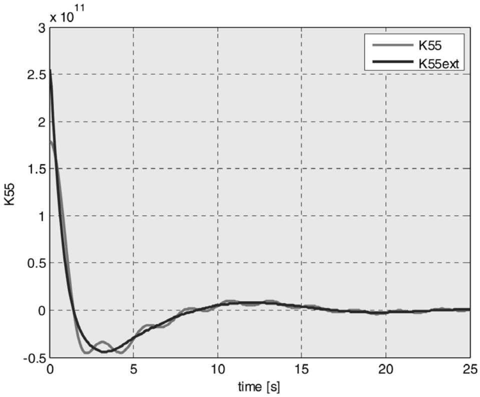

Figure 5. Retardation function 55. Solid–computed from damping without extrapolation. Dashed–computed from damping with extrapolatio ω^{-2}.

FORWARD SPEED CASE

For the case of a constant forward speed case, the Cummins equation (1) becomes

$$[\mathbf{M}+\mathbf{A}(\infty)]\,\ddot{\boldsymbol{\xi}}+\mathbf{B}(\infty)\,\dot{\boldsymbol{\xi}}+\int_0^t \mathbf{K}(t-\tau')\,\dot{\boldsymbol{\xi}}(\tau')\,d\tau'+\mathbf{C}\,\boldsymbol{\xi}=\boldsymbol{\tau} \tag{21}$$

where the constant damping and the retardation function depend on the forward speed U.

The time- and frequency-domain relation revert to

$$\mathbf{A}(\omega)=\mathbf{A}(\infty)-\frac{1}{\omega}\int_0^\infty \mathbf{K}(t)\sin(\omega t)\,dt \tag{22}$$

$$\mathbf{B}(\omega)=\mathbf{B}(\infty)+\int_0^\infty \mathbf{K}(t)\cos(\omega t)\,dt \tag{23}$$

$$\mathbf{K}(t)=\int_0^\infty \left[\mathbf{B}(\omega)-\mathbf{B}(\infty)\right]\cos(\omega t)\,dt \tag{24}$$

$$\mathbf{K}(j\omega)=\mathbf{B}(\omega)-\mathbf{B}(\infty)+j\omega\,[\mathbf{A}(\omega)-\mathbf{A}(\infty)] \tag{25}$$

Therefore, it follows that the results derived in Section 6—as well as the results in the APPENDIX—are valid for the case of constant forward speed modulo substitution $\mathbf{B}(\omega)$ by $\mathbf{B}(\omega)-\mathbf{B}(\infty)$.

CONCLUSION

In this paper, we have considered the consequences the properties of convolution terms in the Cummins' Equation on the high frequency asymptotic values of the added mass and potential damping. We have presented a new derivation of known asymptotic results without using expansions. We have shown that whenever the integral of the potential damping over the frequencies is not zero–as it happens in most practical cases—the hydrodynamic damping and added mass tend to their asymptotic values at a rate of ω^{-2} in the limit as the frequency is very high. The results agree with those previously presented in the literature, but the derivation, however, is believed to be novel, simpler and more elegant. We have also discussed the application to the computation of retardation functions, which can then be used for time-domain system identification of a convolution replacement for simulation of marine structures in waves.

ACKNOWLEDGEMENTS

The authors would like to thank Mr Reza Taghipour, from the Centre for Ship and Ocean Structures (CeSOS) at NTNU–Norway for computing the hydrodynamic data used in the application example with WAMIT.

REFERENCES

Cummins, W., (1962). The impulse response function and ship motions. Schiffstechnik 9 (1661), 101–109.

Damaren, C.J. (2000). Time-domain floating body dynamics by rational approximations of the radiation impedance and diffraction mapping. Ocean Engineering 27, 687–705.

Faltinsen, O.M. (1990) Sea Loads on Ships and Ocean Structures. Cambridge University Press.

Greenhow, M., 1986, High- and low-frequency asymptotic consequences of the Kramers-Kronig relations, J. Eng. Math., 20, 293-306.

Ogilvie, T., 1964. Recent progress towards the understanding and prediction of ship motions. In: 6th Symposium on Naval Hydrodynamics.

Kristiansen, E., A. Hjuslstad and O. Egeland (2005). State-space representation of radiation forces in time-domain vessel models. Ocean Engineering 32, 2195–2216.

Parzen, E. (1954) "Some Conditions for Uniform Convergence of Integrals" Proceedings of the American Mathematical Society, Vol. 5, No. 1. (Feb., 1954), pp. 55-58.

Perez, T. and T. I. Fossen (2006) "Time-domain Models of Marine Surface Vessels Based on Seakeeping Computations." 7th IFAC Conference on Manoeuvring and Control of Marine Vessels MCMC, Portugal, Sept.

Unneland, K., and T. Perez (2007) "MIMO and SISO Identification of Radiation Force Terms for Models of Marine Structures in Waves." IFAC Conference on Control Applications in Marine Systems (CAMS). Bol, Croatia, Sept.

APPENDIX:

Properties of the Convolution Terms in the Cummins Equation.

Low-frequency asymptotic value: $\lim_{\omega \to 0} \mathbf{K}(j\omega) = \mathbf{0}$

The proof of this statement follows from (6):

$$\mathbf{K}(j\omega) = \mathbf{B}(\omega) + j\omega\,[\mathbf{A}(\omega) - \mathbf{A}(\infty)] \ .$$

In the limit as $\omega \to 0$, the potential damping B(ω) $\to$ 0 since there cannot be generated waves (Faltinsen, 1990). Therefore, the real part of K($j\omega$) tends to zero. The imaginary part also tends to zero since the difference A(0) − A(∞) is finite, which follows from (2):

$$\mathbf{A}(0) - \mathbf{A}(\infty) = \lim_{\omega \to 0} \frac{-1}{\omega} \int_0^\infty \mathbf{K}(t) \sin(\omega\, t)\, dt = -\int_0^\infty \mathbf{K}(t) \lim_{\omega \to 0} \frac{\sin(\omega\, t)}{\omega}\, dt = -\int_0^\infty \mathbf{K}(t)\, dt \ ,$$

from which the result follows. Note that regularity conditions are satisfied for the exchange of limit and integration (Parzen, 1954): $f_n(t) = K(t) \sin(2\pi t/n) / (2\pi/n)$ converges uniformly to $f(t) = K(t)$ as $n \to \infty$.

High-frequency asymptotic value: $\lim_{\omega \to \infty} \mathbf{K}(j\omega) = \mathbf{0}$

The proof of this statement also follows from (6):

$$\mathbf{K}(j\omega) = \mathbf{B}(\omega) + j\omega\,[\mathbf{A}(\omega) - \mathbf{A}(\infty)] \ .$$

In the limit as $\omega \to \infty$, the potential damping B (ω) $\to$ 0 since there cannot be generated waves (Faltinsen, 1990). The imaginary part also tends to zero and this follows from (2) and the application of the Riemann-Lebesgue lemma (Ogilvie, 1964):

$$\lim_{\omega \to \infty} \omega[\mathbf{A}(0) - \mathbf{A}(\infty)] = \lim_{\omega \to \infty} \int_0^\infty -\mathbf{K}(t) \sin(\omega\, t)\, dt = \mathbf{0} \ .$$

Initial time value of the retardation function: $\mathbf{K}(t = 0^+) \neq \mathbf{0}$

This follows from (4),

$$\lim_{t \to 0^+} \mathbf{K}(t) = \lim_{t \to 0^+} \frac{2}{\pi} \int_0^\infty \mathbf{B}(\omega) \cos(\omega\, t)\, d\omega = \frac{2}{\pi} \int_0^\infty \mathbf{B}(\omega)\, d\omega \neq \mathbf{0} \, .$$

Note that regularity conditions are satisfied for the exchange of limit and integration (Parzen, 1954); and the last equality follows from energy considerations, which establish that the diagonal terms of B(ω); namely $B_{ii}(\omega) > 0$–see Faltinsen (1990).

Final time value of the retardation function: $\lim_{t\to\infty} \mathbf{K}(t) = \mathbf{0}$

This follows from (4), by application of the Riemann-Lebesgue Lemma:

$$\lim_{t\to\infty} \mathbf{K}(t) = \lim_{t\to\infty} \frac{2}{\pi}\int_0^\infty \mathbf{B}(\omega)\cos(\omega\, t)\, d\omega = \mathbf{0}\,.$$

This property establishes the necessary and sufficient conditions for bounded-input-bounded-output stability of the convolution terms.

Passivity of K($j\omega$): The damping matrix for a structure with zero forward speed and without current is symmetric and positive-semi definite, $B(\omega) = B^T(\omega) \geq 0$ (Faltinsen, 1990), from which it follows the positive realness of K($j\omega$); and thus, the fact that K($j\omega$) is passive. A derivation of passivity in terms of energy functions can be found in Damaren, (2000) and Kristiansen et al (2005). From this it follows that the diagonal elements ($B_{mn}(\omega)$ for terms $m=n$) are positive-semi definite and their integral with respect of ω is different from zero; the integral of the off-diagonal terms (those which are not uniformly zero due to symmetry of the structure), however, could be zero. As commented by Unneland and Perez (2007), the diagonal terms $K_{ii}(j\omega)$are passive, but the off-diagonal terms need only be stable–see previous property.