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A Fuzzy Signature Algebra Framework for Hierarchical Data Aggregation in Marine Monitoring Systems

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ABSTRACT

The algebraic structure of fuzzy signatures with the algebraic product and the algebraic sum operators is investigated. Given a rooted tree $G_S = (V, E)$ and an associated set of aggregation operators A_S, we prove that the family of fuzzy signatures $\mathcal{F}(G_S, A_S)$, equipped with the algebraic product $(*)$ and algebraic sum $(+)$, forms a bounded lattice by verifying the commutativity, the associativity, and the absorption laws. As an application, we demonstrate that this algebraic structure provides a principled and scalable mechanism for hierarchical data aggregation in marine monitoring systems, where sensor uncertainty, incomplete measurements, and multi-level decision hierarchies are effectively managed.

1. Introduction.

Graph theory was introduced by Leonard Euler (1736). A tree represents hierarchical structure in a graphical form. Trees, are started with a root node, and might connect to other nodes, which means that could contain subtrees within them.

A rooted-tree cooperation structure makes it possible to mimic multiple hierarchical structures and splitting processes, which are common organizational forms in a wide range of military, political, and economic activities as well as evolutionary biology. The foundation for creating the fuzzy signature structure is the idea of graph theory.

Zadeh invented the idea of fuzzy sets in 1965 [20]. The concept of fuzzy sets addresses ambiguity and uncertainty. Fuzzy events (Zadeh, 1968) [21], fuzzy automata (Santos, 1972) [14], fuzzy semantics (Zadeh, 1971) [22], and other topics are studied using algebraic product and algebraic sum. In order to explore fuzzy reasoning, which offers a method for handling reasoning problems that are too complicated for exact solutions, algebraic product and algebraic sum are presented.

In 1999, [16] Vamos, Biró, and Kóczy introduced fuzzy signatures as a structured representation allowing signatures to be generated from fuzzy data. In 2004, [19] K.W. Wong and et al. showed the introduction and Fuzzy signature have been applied to medical diagnosis in model complex structured data. In 2012, [13] C. Pozna and et al., document computing, team per-

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formance modeling and classification to the description of the buildings. Structured fuzzy signature frameworks have been developed by modeling complicated systems [1] with hierarchical dependencies using fuzzy automata and their expansions. In the literature, [2] tree-based fuzzy models have been thoroughly examined, especially in relation to fuzzy ω-tree automata, which offer an organized framework for managing hierarchical fuzzy data. In 2020, [9] Laszlo T. Koczya, M. Eugenia Cornejob, and Jesus Medinab proposed the algebraic structure of fuzzy signatures and shown that a family of fuzzy signatures forms a lattice through the use of meet and join operators.

In 2023, [11] introduces a fuzzy inference system–like aggregation operator for fuzzy signatures, allowing hierarchical, rule-based fusion of uncertain information in complex structured data. A vague-intellectual paradigm for assessing environmental supply chain performance under uncertainty was presented by Wang et al. (2023) [17], highlighting the importance of imprecise and ambiguous data in decision-making. Their method emphasizes the value of sophisticated fuzzy-based models for managing intricate environmental systems, which is extremely pertinent to data aggregation in applications involving maritime monitoring. In 2025, [7] this study evaluates the effect of various aggregation operators on fuzzy signature–based robot path planning, highlighting their role in shaping decision accuracy and robustness in uncertain environments.

Complex structured data and interdependent feature issues are addressed by fuzzy signatures. Finding the missing data is another purpose for it. It seeks to determine if a family of fuzzy signatures with suitable algebraic sum and product operations has the algebraic structure of a bounded lattice.

In maritime and coastal areas, marine monitoring devices are essential for tracking environmental factors like temperature, salinity, pH, and pollution levels. Usually, these systems rely on dispersed sensor networks that produce a lot of ambiguous and imprecise data. Variability, noise, and hierarchical system structures continue to be major obstacles to the efficient and trustworthy aggregation of such data.

Fuzzy set theory has been widely used to model uncertainty in such environments. In particular, fuzzy signatures provide a hierarchical representation of complex systems, where data is structured in the form of rooted trees with aggregation operators at internal nodes. However, the algebraic properties of fuzzy signatures and their implications for system-level consistency and stability have not been fully explored in the context of marine monitoring.

In this work, we examine fuzzy signatures' algebraic structure using algebraic product and algebraic sum. We prove that the family of fuzzy signatures forms a bounded lattice under these operations. Furthermore, we demonstrate how this algebraic framework can be effectively applied to hierarchical data aggregation in marine monitoring systems, ensuring consistency, scalability, and robustness in decision-making.

2. Preliminaries.

Definition 2.1. [18] "The graph G is a rooted tree, if it is a tree such that there exists a particular vertex called root such that all edges are oriented so that they move away from the root."

Definition 2.2. [20] "Let V be a set called universe. A fuzzy set X in V is a membership function $\mu_X(X)$ which is to every element $x \in V$ associates a real number from the interval $[0, 1]$ and $\mu_x(X)$ is the grade of membership of x in X."

Definition 2.3. [8] "The **Algebraic Product** of A and B of a fuzzy set X is defined by the membership function

$$\mu_{A*B} = \mu_A * \mu_B$$

for every $A, B \in X$."

Definition 2.4. [8] "The **Algebraic Sum** of A and B of a fuzzy set X is defined by the membership

$$\mu_{A+B} = \mu_A + \mu_B - \mu_A\mu_B$$

for every $A, B \in X$."

Definition 2.5. [3] "A class of signatures is a set of signatures with the same structure. The symbol S will be used to indicate the set of all signatures and the symbol $\hat{S}$ will be used to indicate the set of all signatures trees. A signature is characterized by a structure and by a data collection and a signature tree is characterized only by a structure."

Definition 2.6. [3] "The **signature of a graph**, referred to also as the associated signature, is a bijective function

$$f : T_B \to \hat{S},$$

where T_B is a set of B-trees and $\hat{S}$ is the set of signatures trees."

Definition 2.7. "Let

$$G = (V, E)$$

be a rooted tree with depth $d(G) = 2$ where the set of vertices is

$$V = \{V_0, V_1, V_2, V_{11}, V_{12}, V_{13}, V_{21}, V_{22}\}$$

then set of internal vertices is

$$N = \{V_0, V_1, V_2\}.$$

The root V_0 has two children which are V_1 and V_2. The set of **families of aggregation operators**

$$\{A_3(P), A_2(G)\},$$

where assigned to the internal vertices."

1. $@_v(x, y) = \min\{x, y\}$
2. $@_v\, 1(x) = x$
3. $@_v(x, y, z) = \{x * y * z\}$

Definition 2.8. [9] "A fuzzy signature S associated with $G = (V, E)$, we call **evaluation of the fuzzy signature** S the membership degree assigned to v_0 obtained by executing all the general aggregation operators in S, starting with the membership degrees in the leaves. The evaluation of the fuzzy signature S is denoted as $E(S)$."

Definition 2.9. [9] "The **Algebraic Product of the fuzzy signatures** S_1 and S_2, which will be denoted as $S_1 * S_2$, is the fuzzy signature associated with the rooted tree $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ where $N_{G_1 \cap G_2}$ denotes the set of internal vertices and $L_{G_1 \cap G_2}$ the set of leaves. The membership degrees assigned to every leaf in $L_{G_1 \cap G_2}$ and the aggregation operators assigned to every internal vertex in $N_{G_1 \cap G_2}$ will be given by the following rules:"

1. If $v \in N_{G_1 \cap G_2}$, the aggregation operator assigned to v is:

$$@_v = \inf \left\{ @_v^{(1)},\ @_v^{(2)} \right\}.$$

2. If $v \in L_{G_1 \cap G_2}$, the membership degree assigned to v is:

$$\mu_v = \inf \left\{ \mu_v^{(1)},\ \mu_v^{(2)} \right\}.$$

3. Theorem.

Theorem 1. *Let G_S be the rooted tree and A_S be the set of families of aggregation operators associated to the fuzzy signature S. The family of fuzzy signature $F(G_S, A_S)$ and the operators Algebraic Product $*$ and Algebraic Sum $+$ together to form a bounded lattice.*

Proof. Consider three rooted trees $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$ where M_1, M_2 and M_3 are the set of internal vertices and F_1, F_2 and F_3 are the set of leaves for the rooted tree G_1, G_2 and G_3 respectively. Let S_1, S_2 and S_3 are the fuzzy signatures of $F(G_S, A_S)$ associated with the rooted trees.

Demonstration on $(F(G_S, A_S), *, +)$ to lattice,

(i) Commutativity law:

$$S_1 * S_2 = S_2 * S_1 \quad \text{and} \quad S_1 + S_2 = S_2 + S_1$$

(ii) Associativity law:

$$(S_1 * S_2) * S_3 = S_1 * (S_2 * S_3)$$

and

$$(S_1 + S_2) + S_3 = S_1 + (S_2 + S_3)$$

(iii) Absorption law:

$$S_1 * (S_1 + S_2) = S_1$$

and

$$S_1 + (S_1 * S_2) = S_1$$

- **Commutativity:** Consider two fuzzy signatures S_1 and S_2 from the family of fuzzy signatures. By the commutativity law for algebraic product, it is related to the intersection for both the fuzzy signatures S_1 and S_2 [is given in Figure 1 and Figure 2]. The set of aggregation operators $\{A_2(G), A_1\}$ were allocated to the internal vertices [as provided in Figure 3] in order to guarantee the commutativity law.

$$A_2(G) = \min\{0.7, 0.2\} = 0.2$$

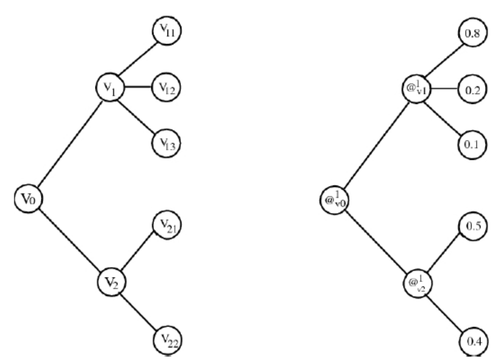
$$A_1 = 1(0.1) = 0.1$$

The algebraic product of $S_1 * S_2$ is,

$$\begin{aligned} \mu_{A*B} &= \mu_A * \mu_B \\ &= 0.2 * 0.1 \\ &= 0.02 \end{aligned} \tag{1}$$

Similarly, the algebraic product of $S_2 * S_1$ is 0.02. (2)

Figure 1: *Left - Rooted tree G_1; Right - fuzzy signature S_1.*



Source: Authors.

From (1) and (2),

$$S_1 * S_2 = S_2 * S_1 \tag{I}$$

By the commutative law for algebraic sum, it is related to the union for both the fuzzy signatures S_1 and S_2.

To ensure the commutativity law, the set of aggregation operators families $\{A_3(P), A_2(G)\}$, where assigned to the internal vertices [is given in Figure 4].

$$A_3(P) = \{0.8 * 0.4 * 0.1\} = 0.032$$

$$A_2(G) = \min\{0.5, 0.4\} = 0.4$$

The algebraic sum of $S_1 + S_2$ is,

$$\begin{aligned} \mu_{A+B} &= \mu_A + \mu_B - \mu_A \mu_B \\ &= 0.032 + 0.4 - (0.032)(0.4) \\ &= 0.4192 \end{aligned} \tag{3}$$

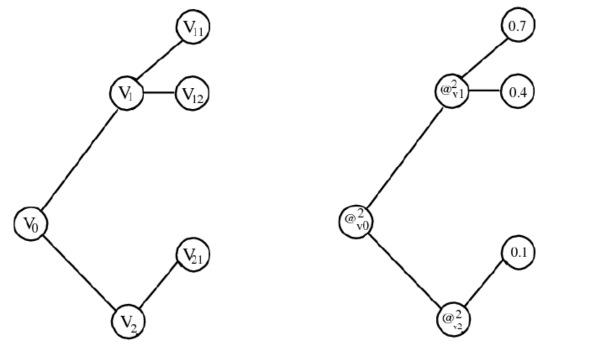
Similarly, the algebraic sum of $S_2 + S_1$ is 0.4192. (4)

From (3) and (4),

$$S_1 + S_2 = S_2 + S_1 \tag{II}$$

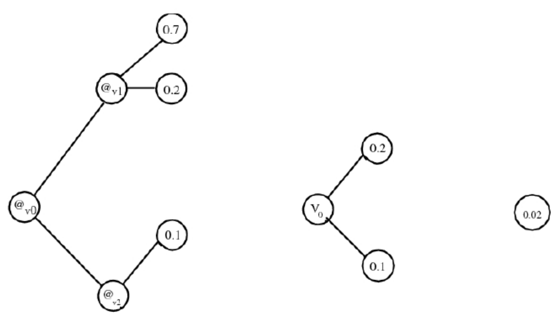
Therefore, from (I) and (II), commutativity law for algebraic product and sum is verified.

Figure 2: *Left - Rooted tree G_1; Right - fuzzy signature S_1.*



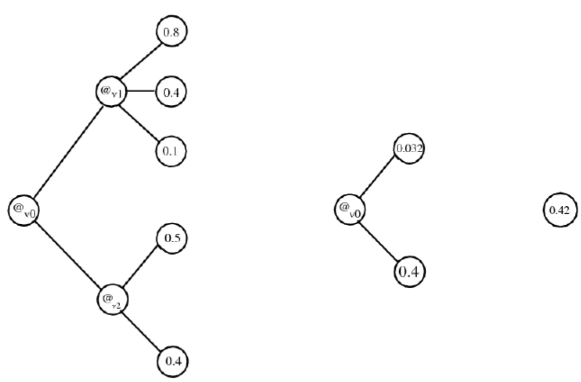
Source: Authors.

Figure 3: *Evaluation of fuzzy signature $S_1 * S_2$.*



Source: Authors.

Figure 4: *Evaluation of fuzzy signature $S_1 + S_2$.*



Source: Authors.

- **Associativity:** Consider the rooted tree $(G_1 * G_2) * G_3$ and $G_1 * (G_2 * G_3)$ to obtain the fuzzy signatures $(S_1 * S_2) * S_3$ and $S_1 * (S_2 * S_3)$ where Algebraic product $(*)$ is related to the intersection $(\cap)$.

Let the set of vertices which consists of the set of internal vertices and the set of leaves and are denoted by $M_{G_1 \cap G_2 \cap G_3}$ and $F_{G_1 \cap G_2 \cap G_3}$ in $(S_1 * S_2) * S_3$ and in $S_1 * (S_2 * S_3)$. Now, we construct a Fuzzy Signature by the association of vertices, membership degrees and aggregation operators.

The rules for associativity is stated here:

(i) Using aggregation operators in the Internal Vertices (M)

1. If $v \in M_1$, $v \in M_2$ and $v \in M_3$, then we arrive two possibilities that $v \in F_{G_1 \cap G_2 \cap G_3}$ and $v \in M_{G_1 \cap G_2 \cap G_3}$. Then v has 0 value in $(S_1 * S_2) * S_3$ and in $S_1 * (S_2 * S_3)$ when there is no common vertices for the three aggregators in the three fuzzy signatures.
2. The Internal vertices are assigned by an aggregator operator, they are named to be $@_v^1$, $@_v^2$ and $@_v^3$, where $@_v^1$ assigned to $v \in M_1$ in S_1, $@_v^2$ assigned to $v \in M_2$ in S_2 and $@_v^3$ assigned to $v \in M_3$ in S_3. By using,

$$@_v = \inf\left\{@_v^1, @_v^2\right\}.$$

Therefore, the operator assigned to $v \in M_{G_1 \cap G_2 \cap G_3}$ in $(S_1 * S_2) * S_3$ is,

$$\inf\left\{\inf\left\{@_v^1, @_v^2\right\}, @_v^3\right\} \tag{5}$$

Again applying the operator assigned to $v \in M_{G_1 \cap G_2 \cap G_3}$ in $S_1 * (S_2 * S_3)$ is,

$$\inf\left\{@_v^1, \inf\left\{@_v^2, @_v^3\right\}\right\} \tag{6}$$

From 5 and 6, using the associativity we get,

$$\inf\left\{\inf\left\{@_v^1, @_v^2\right\}, @_v^3\right\} = \inf\left\{@_v^1, \inf\left\{@_v^2, @_v^3\right\}\right\} \tag{III}$$

(ii) Using Membership degree in the Leaves (F)

Membership degrees μ_v^1, μ_v^2 and μ_v^3 are assigned to $v \in F_1$ in S_1, $v \in F_2$ in S_2 and $v \in F_3$ in S_3.

1. If $v \in F_1$, $v \in M_2$ and $v \in M_3$ then $v \in F_{G_1 \cap G_2 \cap G_3}$. When $v \in F_1$ in S_1, we assign the membership degree μ_v^1, $v \in M_2$ in S_2, we assign the membership degree $@_v^2$, $v \in M_3$ in S_3, we assign the membership degree $@_v^3$, then the membership degree assigned to $v \in F_{G_1 \cap G_2 \cap G_3}$ in $S_1 * (S_2 * S_3)$ is μ_v^1.
2. If $v \in F_1$, $v \in F_2$ and $v \in M_3$ then $v \in F_{G_1 \cap G_2 \cap G_3}$. Let μ_v^1, μ_v^2 and $@_v^3$ be the membership degree and the aggregator operator are assigned. Then the membership degree assigned to $v \in F_{G_1 \cap G_2 \cap G_3}$ in $(S_1 * S_2) * S_3$ combine with the membership degree in $S_1 * (S_2 * S_3)$. This implies,

$$\inf\{\mu_v^1, \mu_v^2\}.$$

3. If $v \in F_1$, $v \in F_2$ and $v \in F_3$ then $v \in F_{G_1 \cap G_2 \cap G_3}$, where the membership degrees μ_v^1, μ_v^2 and μ_v^3 are assigned to $v \in F_1$ in S_1, $v \in F_2$ in S_2 and $v \in F_3$ in S_3.
   Finally, for $v \in F_{G_1 \cap G_2 \cap G_3}$ in $(S_1 * S_2) * S_3$ the membership degree is assigned to be,
   $$\inf\left\{\inf\{\mu_v^1, \mu_v^2\}, \mu_v^3\right\} \tag{7}$$
   Again for $v \in F_{G_1 \cap G_2 \cap G_3}$ in $S_1 * (S_2 * S_3)$ the membership degree is assigned to be,
   $$\inf\left\{\mu_v^1, \inf\{\mu_v^2, \mu_v^3\}\right\} \tag{8}$$
   From 7 and 8, using the associativity we get,
   $$\inf\left\{\inf\{\mu_v^1, \mu_v^2\}, \mu_v^3\right\} = \inf\left\{\mu_v^1, \inf\{\mu_v^2, \mu_v^3\}\right\} \tag{IV}$$
   Therefore, from III and IV we arrive at a conclusion that the associativity for algebraic product holds. Similarly we can prove the associativity for the algebraic sum, by using Supremum.

- **Absorption:** If we consider the set of internal vertices $M_{G_1 \cap (G_2 \cup G_3)}$ and the set of leaves $F_{G_1 \cap (G_2 \cup G_3)}$ for the fuzzy signature $S_1 * (S_1 + S_2)$ by using $V_1 \cap (V_2 \cup V_3) = V_1$, then it is easy to show that $S_1 * (S_1 + S_2) = S_1$.

  The rules for absorption are stated here:

  1. If $v \in M_1$ and $v \in M_2$ then $v \in M_{G_1 \cap (G_2 \cup G_3)}$. The aggregator operators $@_v^1$ and $@_v^2$ are assigned to the internal vertices, that is, $v \in M_1$ in S_1 and $v \in M_2$ in S_2. We can assign the aggregator operator to $v \in M_{G_1 \cap (G_2 \cup G_3)}$ in $S_1 * (S_1 + S_2) = S_1$,
     $$\inf\left\{@_v^1, \sup\left\{@_v^2, @_v^3\right\}\right\} = @_v^1$$
  2. If $v \in F_1$ and $v \in F_2$ then $v \in F_{G_1 \cap (G_2 \cup G_3)}$. The membership degrees μ_v^1 and μ_v^2 are assigned to the leaves, that is, $v \in F_1$ in S_1 and $v \in F_2$ in S_2. We can assign the membership degree to $v \in F_{G_1 \cap (G_2 \cup G_3)}$ in $S_1 * (S_1 + S_2) = S_1$,
     $$\inf\left\{\mu_v^1, \sup\left\{\mu_v^2, \mu_v^3\right\}\right\} = \mu_v^1$$
  3. If $v \in M_1$ and $v \in F_2$ then $v \in M_{G_1 \cap (G_2 \cup G_3)}$. The aggregator operator $@_v^1$ is assigned to the internal vertices $v \in M_1$ in S_1 and the membership degree μ_v^1 is assigned to the leaves $v \in F_2$ in S_2. Again, we can assign the aggregator operator to $v \in M_{G_1 \cap (G_2 \cup G_3)}$ in $S_1 * (S_1 + S_2) = S_1$,
     $$\inf\left\{@_v^1, @_v^2\right\} = @_v^1$$
  4. If $v \in F_1$ and $v \notin V_2$ then $v \in F_{G_1 \cap (G_2 \cup G_3)}$. The membership degree μ_v^1 is assigned to the leaves $v \in F_1$ in S_1. We can assign the membership degree to $v \in F_{G_1 \cap (G_2 \cup G_3)}$ in $S_1 * (S_1 + S_2) = S_1$,
     $$\inf\left\{\mu_v^1, \mu_v^2\right\} = \mu_v^1$$
  5. If $v \in M_1$ and $v \notin V_2$ then $v \in M_{G_1 \cap (G_2 \cup G_3)}$. The aggregator operator $@_v^1$ is assigned to the internal vertices $v \in M_1$ in S_1. Again, we can assign the aggregator operator to $v \in M_{G_1 \cap (G_2 \cup G_3)}$ in $S_1 * (S_1 + S_2) = S_1$,
     $$\inf\left\{@_v^1, @_v^2\right\} = @_v^1$$

  From all the above rules, we obtain the same $v \in S_1$. Therefore, we conclude that
  $$S_1 * (S_1 + S_2) = S_1.$$
  Similarly, the other case can also be proved. Therefore, the given fuzzy signature S from the family of fuzzy signatures $F(G_S, A_S)$ attains the least and the greatest element.

  Hence,
  $$(F(G_S, A_S), *, +)$$
  is a bounded lattice.

4. Framework for Marine Monitoring Systems.

Marine monitoring systems consist of distributed sensors deployed in oceanic and coastal environments to measure parameters such as temperature, salinity, pH, and pollution levels. These systems are inherently hierarchical, where raw sensor data is collected at lower levels and progressively aggregated at higher levels before reaching a central decision unit. Due to uncertainty, noise, and incomplete information, an efficient aggregation mechanism is required.

In this section, we propose a framework based on fuzzy signature algebra for hierarchical data aggregation in marine monitoring systems.

4.1. Hierarchical Modeling Using Fuzzy Signatures.

A marine monitoring system can be naturally modeled as a rooted tree
$$G_S = (V, E)$$
where:

- **Leaf nodes** represent sensor measurements (e.g., temperature, salinity, pH).
- **Internal nodes** represent aggregation units (e.g., cluster heads or processing units).
- **The root node** represents the final decision output.

Each leaf node $v \in F$ is associated with a membership degree
$$\mu_v \in [0, 1],$$
which indicates the degree of satisfaction of a particular environmental condition.

Each internal node is associated with an aggregation operator that combines the incoming values from its child nodes. Importantly, different internal nodes may employ *different* operators at different levels of the hierarchy (heterogeneous aggregation), which is a key advantage of the fuzzy signature framework over standard hierarchical fuzzy systems.

In the case of sensor failure and it reported no data, the default value $\mu_v = 0.5$ is assigned to the leaf node corresponding to it, which avoids the false hazard ($\mu_v = 0$) or the false safety assignments. ($\mu_v = 0$).

5. Aggregation Using Algebraic Product and Sum

Consider S_1 and S_2 to be the fuzzy signatures representing the two sets of sensor observations.

- **Algebraic Product (Conservative Aggregation):** When all the conditions must be satisfied simultaneously, algebraic product can be used. It is defined as:

$$\mu_{S_1 * S_2} = \mu_{S_1} \cdot \mu_{S_2}.$$

  The algebraic product changes the aggregate value quickly when any one of the factors deviates from the target value, making it perfect for *cluster-to-root aggregation*, where all cluster conditions must be satisfied.

- **Algebraic Sum (Flexible Aggregation):** When at least one condition contributes to the decision, the algebraic sum is used. It is defined as:

$$\mu_{S_1 + S_2} = \mu_{S_1} + \mu_{S_2} - \mu_{S_1}\mu_{S_2}.$$

  This method is aligned for *leaf-to-cluster aggregation*, where partial evidence is contributed.

  Thus, the two aggregations form a complementary pair used for different levels of the monitoring hierarchy:

  - Cluster → Root: Algebraic product can be applied.
  - Leaf → Cluster: Algebraic sum can be applied.

- **Weighted Extension:** The aggregation operators can be extended using weighted relevance, where $w_i \in [0, 1]$ is the stability weight of the i-th sensor, to account for variations in sensor reliability and priority:

$$\mu^{(w)}_{S_1 * S_2} = \prod_i \mu_i^{w_i}.$$

5.1. Illustrative Example.

Let us consider a marine monitoring system. The system has three sensors measuring:

- Temperature: $\mu_1 = 0.7$
- pH level: $\mu_2 = 0.5$
- Salinity: $\mu_3 = 0.4$

Using Algebraic Product

$$\mu = 0.7 \times 0.5 \times 0.4 = 0.14.$$

This calculation shows a strict evaluation, where all parameters must satisfy the condition together.

Using Algebraic Sum

For three membership values, the algebraic sum follows the probabilistic inclusion–exclusion principle:

$$\mu = \mu_1 + \mu_2 + \mu_3 - (\mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3) + \mu_1\mu_2\mu_3.$$

Then,

$$\begin{aligned}\mu &= (0.7 + 0.5 + 0.4) - (0.7 \times 0.5 + 0.5 \times 0.4 + 0.7 \times 0.4)\\ &\quad + (0.7 \times 0.5 \times 0.4)\\ &= 1.6 - (0.35 + 0.20 + 0.28) + 0.14\\ &= 1.6 - 0.83 + 0.14\\ &= 0.91.\end{aligned}$$

Equivalently, using the complement formulation:

$$\begin{aligned}\mu &= 1 - (1 - 0.7)(1 - 0.5)(1 - 0.4)\\ &= 1 - (0.3)(0.5)(0.6) = 1 - 0.09 = 0.91.\end{aligned}$$

Table 1: Aggregation results for the illustrative example ($\mu_1 = 0.7, \mu_2 = 0.5, \mu_3 = 0.4$).

Operation	Formula	Result
Algebraic Product	$0.7 \times 0.5 \times 0.4$	0.14
Algebraic Sum	$1 - (0.3)(0.5)(0.6)$	0.91

5.2. Role of Bounded Lattice Structure.

The family of fuzzy signatures forms a bounded lattice under algebraic product and sum, according to Theorem 1. In marine monitoring systems, this offers the following benefits:

- **Commutativity and Associativity:** The algebraic operations are commutative and associative, guaranteeing that the order in which sensor inputs are processed has no role on the aggregation results.
- **Stability:** During hierarchical aggregation, the absorption law guarantees that no contradicting results exist.
- **Scalability:** Large sensor networks are supported by hierarchical tree topologies; given a tree with n leaf nodes, aggregation is finished in $O(n)$ time, indicating computational feasibility.
- **Robustness:** The framework effectively manages uncertainty in sensory data, including instances of partial or missing measurements. While the largest element 1 symbolizes complete satisfaction, the lowest element 0 shows the total absence of condition satisfaction.

6. System Interpretation.

Consequently, the suggested framework permits:

- Heterogeneous, level-specific aggregating operators are supported in the structured representation of marined data using fuzzy signatures.
- Reliable aggregation with the option to include sensor reliability weights through algebraic operations.
- Consistency throughout the bounded lattice structure is guaranteed mathematically.
- Handling sensory failures with grace and minimizing uncertainty. Because of this, the architecture is appropriate for real-world maritime surveillance applications using partial, hierarchical, and uncertain data.

Conclusions.

Fuzzy signature plays a viral role in structuring data into vectors of fuzzy values, each of which can be further vector are introduced in handling complex structured data and problems with interdependent features. The concept of algebraic product and sum was applied in the algebraic structure of fuzzy signature. The family of fuzzy signatures form a bounded lattice based on algebraic product and sum is determined. The framework's applicability to actual marine environments is further strengthened by useful improvements including heterogeneous operator assignment, weighted sensor reliability, and sensor failure imputation. Adaptive weight learning and validation against benchmark oceanic datasets will be the main topics of future research.

References.

[1] Cokilavany, R., Venkatesan, R. and Kavikumar, J., (2015). *Fuzzy ω-automata and Its Relationships*, International Journal of Pure and Applied Mathematics, Vol. 99, No. 1, 23–36.

[2] Cokilavany, D. and Venkatesan, R., (2016). *Fuzzy ω-Tree Automata*, Annals of Pure and Applied Mathematics, Vol. 11, No. 1, 39–44.

[3] Claudiu Pozna, Nicuşor Minciulete, Radu-Emil Precup, László T. Kóczy and Áron Balagig. (2012). *Signatures: Definitions, operators and applications to fuzzy modelling*, Fuzzy Sets and Systems, 201, 86–104.

[4] Davey, B. and Priestley, H., (2002). *Introduction to Lattices and Order*, Second Edition, Cambridge University Press.

[5] Gaines, B. R. and Kohout, L. J. (1977). *The fuzzy decade: A bibliography of fuzzy systems and closely related topics*, International Journal of Man-Machine Studies, Vol. 9.

[6] Kandel, A. and Byatt, W. J. (1978). *Fuzzy sets, fuzzy algebra, and fuzzy statistics*, Proceedings of the IEEE, Vol. 66.

[7] Karadeniz, A. M., Hajdu, C., Ballagi, Á., and Kóczy, L. T. (2025). Evaluating the Impact of Aggregation Operators on Fuzzy Signatures for Robot Path Planning. *Sensors*, 25(23), 7342.

[8] Klir, G. J. and Yuan, B., (1995). *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall.

[9] Laszlo T. Koczya, EugeniaCornejob M., JesusMedinab (2020). *Algebraic structure of fuzzy signatures*, Fuzzy Sets and Systems, Elsevier.

[10] Li I.H., Lee I.W., (2011). *A hierarchical structure of observer-based adaptive fuzzy-neural controller for MIMO systems*, Fuzzy Sets and Systems, Vol. 1, pp. 52–82.

[11] Lilik, F., Bukovics, Á., Kóczy, L.T. (2023). Fuzzy Inference System-like Aggregation Operator for Fuzzy Signatures. In: Cornejo, M.E., Harmati, I.Á., Kóczy, L.T., Medina-Moreno, J. (eds). *Computational Intelligence and Mathematics for Tackling Complex Problems 4*. Studies in Computational Intelligence, Vol. 1040. Springer, Cham.

[12] Masaharu Mizumoto, Kokichi Tanaka, (1981). *Fuzzy sets and their operators*, Information and Control, Vol. 48, pp. 30–48.

[13] Pozna C., Minculete N., Precup R. E., Koczy L.T., Ballagi A., (2012). *Signatures: definitions, operators and applications to fuzzy modelling*, Fuzzy Sets and Systems, pp. 86–104.

[14] Santos E. S., (1972). Max-product machines, *Journal of Mathematical Analysis and Applications*, Vol. 23.

[15] Takahiro Matsuda, Kenta Takahashi and Takao Murakami, (2016). *Fuzzy Signatures: Relaxing Requirements and a New Construction*.

[16] Vamos L.T.K.T., Gy. Biro, (1999). *Fuzzy signatures*, in: SIC-EUROFUSE'99, Budapest, pp. 210–217.

[17] Wang, J., Prabhu, M., Sewagegn, A.A. and Sahu, A.K., (2023). *Environmental Supply Chain Performance Mapping of an Enterprise by Exploring a Novel Vague-Intellectual Approach*, Discrete Dynamics in Nature and Society, 1, p. 2133346.

[18] West, D. B., (2001). *Introduction to Graph Theory*, 2nd ed., Prentice Hall.

[19] Wong K.W., Gedeon T., Koczy L., (2004). *Construction of fuzzy signature from data: an example of SARS preclinical diagnosis system*, in: 2004 IEEE International Conference on Fuzzy Systems (IEEE Cat. No. 04CH37542), Vol. 3, pp. 1649–1654.

[20] Zadeh L.A., (1965). *Fuzzy sets*, Information and Control, Vol. 8, pp. 338–353.

[21] Zadeh L. A., (1968). *Probability measures of fuzzy events*, Journal of Mathematical Analysis and Applications, Vol. 23.

[22] Zadeh L. A., (1971). *Quantitative fuzzy semantics*, Information Sciences, Vol. 3.